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# Control Small-Signal Characterization of Perturbed Geostationary Satellite Orbit

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***Abstract*— This paper gives the theoretical control small-signal description of perturbed Keplarian satellite orbit. Starting from a pair of second-order differential equations obtained previously based on Newton's law of motion, a control state space description is formulated. It is shown that the application of Laplace transform to the system, input, and output matrices leads the transfer function of satellite orbit. Consideration of small signal variation in the input and output matrices gives the control characterization of the satellite orbit.**

***Keywords*— Keplarian orbit; state-space description; control small-signal; perturbation; transfer function**

## I. Introduction

Natural or artificial body moving around a celestial body such as planets and stars as depicted in Fig. 1, [1], are well-studied. Many subtle effects such as earth's oblateness, solar and lunar effects, solar radiation pressure perturb earth satellite orbits, invalidating the simple orbits predicted by two-body gravity equations [2], [3]. Therefore, a study of nonlinear model of geostationary communication satellite will provide an insight to the orbital parameters which ensure stability under these subtle effects.

In the seminar approach presented in [4], or its variants [5], [6], the emphasis is on small eccentricities and inclinations. This has made [4]'s widely embraced closed form solution of the earth's artificial satellite problem to be a significant part of SGP4. SGP4 features continuous availability of matching orbital elements. SGP4 is an analytical orbit model for Low-Earth orbiting satellites that is widely used for the propagation of North American Aerospace Defense Command (NORAD) two-line elements. Two-line elements may hence be generated completely independent of NORAD. It is observed that their use as exclusive source of orbital information simplifies the operations concept and reduces mission costs through the extensive use of existing low-cost mission support software.

However, the understanding noted in [7], that is, proceeding by averaging is not the unique possibility in obtaining analytical solutions by general perturbations, and the use of intermediary orbits like useful approximate solutions of the problem of artificial satellite theory (AST) is proposed.

In the study of artificial satellites, perturbations are considered to be limited to the effects of the second zonal harmonic [5].

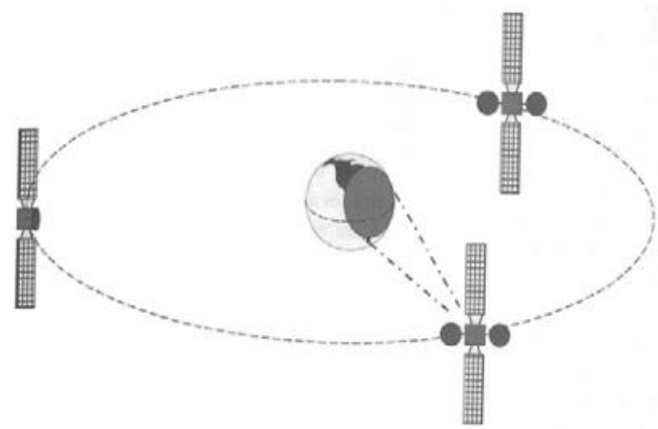

Fig. 1 A system of three geostationary communication satellites provides nearly worldwide coverage [1].

The idea of AST is developed further such that AST intermediaries are obtained by reorganizing the terms of the disturbing function, a simple expedient that may be preceded by the simultaneous addition and subtraction to the geo-potential of some smartly-chosen supplementary terms. Reference [8] shows that after reorganization, a part of the Hamiltonian that admits a separable generating function is taken as the (zero order) integrable problem whereas the rest of the disturbing function is taken as the perturbation, which may be further neglected. Hence, common intermediaries are formulated in the same variables as the original satellite problem —traditionally in spherical variables.

Therefore, it is demonstrated in [8] that the intermediary solution provides an efficient alternative for the analytical propagation of low earth orbits in a range determined by non-impact orbits with eccentricities below one tenth.

In this work, it is considered that if the system operates around an equilibrium point and if the signals involved are small signals, then it is possible to approximate the non-linear system by a linear system. Such a linear system is equivalent to the nonlinear system considered within a limited operating range.

Such a linearized model (linear, time-invariant model) is very important in control aspect of satellite orbit.

The remaining part of this work is organized as follows. Discussion on control systems features in section II. The formulation of control small-signal expression is presented in section III. Section IV provides discussion with respect to the work presented in this paper. The conclusion about this work is drawn in section V.

## II. Control System

### A. *Classical Control System*

At a simplified level, the systems or plants that can be considered by using classical control ideas are linear and time invariant, and have a single input and a single output [9], [10]. The primary aim of the designer using classical control design methods is to stabilize a plant, whereas secondary aims may involve obtaining a certain transient response, bandwidth, disturbance rejection, steady state error, and robustness to plant variations or uncertainties. The designer's methods are a combination of analytical ones (e. g., Laplace transform, Routh test), graphical ones (e.g., Nyquist plots, Nichols charts), and a good deal of empirically based knowl- edge (e. g., a certain class of compensator works satisfactorily for a certain class of plant). For higher-order systems, multiple-input systems, or systems that do not possess the properties usually assumed in the classical control approach, the de- signer's ingenuity is generally the limiting factor in achieving a satisfactory design.

### B. *Modern Control System*

Two of the main aims of modern, as opposed to classical, control are to de-empiricise control system design and to present solutions to a much wider class of control problems than classical control can tackle. One of the major ways modern control sets out to achieve these aims is by providing an array of analytical design procedures that facilitate the design task.

Modern control theory is contrasted with conventional control theory in that the former is applicable to multiple-input-multiple-output systems, which may be linear or nonlinear, time invariant or time varying, while the latter is applicable only to linear time-invariant single-input-single-output systems. Also, modern control theory is essentially a time-do- main approach, while conventional control theory is a complex frequency-domain approach [11].

The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least n variables $x_1, x_2, x_3, \dots x_n$ are needed to completely describe the behavior of a dynamic system (so that once the input is given for $t \geq t_o$ to and the initial state at $t = t_o$ is specified, the future state of the system is completely determined), then such n variables are a set of state variables.

State-space representation is the design method employed in this work. State-space analysis is concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, out- put variables, and state variables. The state-space representation for a given system is not unique, except that the number of state variables is the same for any of the different state-space representations of the same system [11].

The dynamic system must involve elements that memorize the values of the input for $t \geq t_1$ . Since integrators in a continuous-time control system serve as memory devices, the outputs of such integrators can be considered as the variables that define the internal state of the dynamic system. Thus the outputs of integrators serve as state variables. The number of state variables to completely define the dynamics of the system is equal to the number of integrators involved in the system [11].

## III. Small-Signal Consideraton of satellite orbit

Starting from Newton's law of motion, the satellite motion can be described using the following second order differential equations [12].

$$\frac{d^2 r_o}{dt^2} - r_o\left(\frac{d\beta_o}{dt}\right)^2 - \frac{f_{r_o}}{m} = -\frac{\mu}{r_o^2} \tag{1}$$

$$r_o\left(\frac{d^2\beta_o}{dt^2}\right) + 2\left(\frac{dr_o}{dt}\right)\left(\frac{d\beta_o}{dt}\right) - \frac{f_{\beta_o}}{mr_o} = 0 \tag{2}$$

Rearranging (1) and (2) we get

$$\frac{d^2 r_o}{dt^2} = r_o\left(\frac{d\beta_o}{dt}\right)^2 + \frac{f_{r_o}}{m} - \frac{\mu}{r_o^2} \tag{3}$$

$$\left(\frac{d^2\beta_o}{dt^2}\right) = -2\left(\frac{dr_o}{dt}\right)\left(\frac{d\beta_o}{dt}\right)\left(\frac{1}{r_o}\right) + \frac{f_{\beta_o}}{mr_o^2} \tag{4}$$

$$\frac{GM_E m}{R_O^2} = mR_o\omega^2 \tag{5}$$

$$GM_E = \mu = \omega^2 R_o^3 \tag{6}$$

The expected geostationary orbit is described as

$$r_o = R_o \tag{7}$$

$$\beta_o = \beta + \omega t \tag{8}$$

Where $r_o$ is the actual radius of orbit from the centre of the earth, G is universal gravitational constant, ω is angular velocity, $R_o$(constant) is the desired radius of orbit $\beta_o$ is sweep angle, m is mass of satellite, $M_E$ is mass of earth, $f_{r_o}$ and $f_{\beta_o}$ are the satellite propulsion in $r_o$ and $\beta_o$ directions.

Assuming $x_1, x_2, x_3, x_4$ are state variables defined as follows:

$$x_1 = r_o - R_o \tag{9}$$

$$x_2 = \frac{dr_o}{dt} \tag{10}$$

$$x_3 = \beta_o - (\beta + \omega t) \tag{11}$$

$$x_4 = \frac{d\beta_o}{dt} - \omega \tag{12}$$

Consider the differential with respect to time of (1), (2), (3) and (4) as follow

$$\frac{dx_1}{dt} = \dot{x}_1 = \frac{dr_o}{dt} = x_2 \tag{13}$$

$$\frac{dx_2}{dt} = \dot{x}_2 = \frac{d^2 r_o}{dt^2} = (x_1 + R_o)(x_4 + \omega)^2 - \frac{\mu}{(x_1 + R_O)^2} + \frac{f_r}{m} \tag{14}$$

$$\frac{dx_2}{dt} = \frac{(x_1+R_o)^3(x_4+\omega)^2-\mu}{(x_1+R_o)^2} + \frac{f_r}{m} \quad (15)$$

Using Binomial theorem, (15) becomes

$$\frac{dx_2}{dt} = \frac{(R_o^3+3x_1R_o^2)(\omega^2+2\omega x_4)-\omega^2R_o^3}{R_o^2+2R_ox_1} \quad (16)$$

$$\frac{dx_2}{dt} = 3\omega^2 + 2R_o\omega x_4 + \frac{f_{r_o}}{m} \quad (17)$$

$$\frac{dx_3}{dt} = \dot{x}_3 = \frac{d\beta_o}{dt} - \omega = x_4 \quad (18)$$

$$\frac{dx_4}{dt} = \dot{x}_4 = -\frac{2}{r_o}\left(\frac{dr_o}{dt}\right)\left(\frac{d\beta_o}{dt}\right) + \frac{f_{\beta_o}}{mr_o} \quad (19)$$

$$\frac{dx_4}{dt} = \dot{x}_4 = -2\frac{x_2(x_4+\omega)}{(x_1+R_o)} + \frac{f_{\beta_o}}{m(x_1+R_o)} \quad (20)$$

$$\dot{x}_4 = \frac{-2\omega x_2\left(1+\frac{x_4}{\omega}\right)}{R_o\left(1+\frac{x_1}{R_o}\right)} + \frac{f_{\beta_o}}{mR_o\left(1+\frac{x_1}{R_o}\right)} \quad (21)$$

$$\dot{x}_4 = \frac{-2\omega x_2}{R_o} + \frac{f_{\beta_o}}{mR_o} \quad (22)$$

$$\dot{x}(t) = Ax(t) + Bu(t) \ state\ equation \quad (23)$$

$$y(t) = Cx(t) + Du(t) \ output\ equation \quad (24)$$

Where, $A$ is the coefficient matrix of the geostationary satellite orbit, $B$ is the driving matrix, $C$ is the output matrix, and $D$ is the transmission matrix.

If the system perturbation vector is defined as $\left[f_{r_o}, f_{\beta_o}\right]^T$, then (13), (17), (18) and (22) can be expressed in the state variable format of (23).

$$\begin{bmatrix}\dot{x_1}(t)\\ \dot{x_2}(t)\\ \dot{x_3}(t)\\ \dot{x_4}(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 3\omega^2 & 0 & 0 & 2R_o\omega\\ 0 & 0 & 0 & 1\\ 0 & -\frac{2\omega}{R_o} & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{1}{m} & 0\\ 0 & 0\\ 0 & \frac{1}{mR_o}\end{bmatrix}\begin{bmatrix}U_1(t)\\ U_2(t)\end{bmatrix} \quad (25)$$

$$\left[\dot{X}(t)\right] = \begin{bmatrix}0 & 1 & 0 & 0\\ 3\omega^2 & 0 & 0 & 2R_o\omega\\ 0 & 0 & 0 & 1\\ 0 & -\frac{2\omega}{R_o} & 0 & 0\end{bmatrix}[X(t)] + \begin{bmatrix}0 & 0\\ \frac{1}{m} & 0\\ 0 & 0\\ 0 & \frac{1}{mR_o}\end{bmatrix}[U(t)] \quad (26)$$

From (26)

$$A = \begin{bmatrix}0 & 1 & 0 & 0\\ 3\omega^2 & 0 & 0 & 2R_o\omega\\ 0 & 0 & 0 & 1\\ 0 & -\frac{2\omega}{R_o} & 0 & 0\end{bmatrix} \quad (27)$$

$$B = \begin{bmatrix}0 & 0\\ \frac{1}{m} & 0\\ 0 & 0\\ 0 & \frac{1}{mR_o}\end{bmatrix} \quad (28)$$

$$y = x_3 \quad (29)$$

$$C = [0 \quad 0 \quad 1 \quad 0] \quad (30)$$

Equation (26) is solved by using Laplace transform method as follows. From (23), i.e

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (31)$$

Taking the Laplace transform of (31), we get

$$sX(s) - x(0) = AX(s) + BU(s) \quad (32)$$

Where x(0) is the initial condition

$$(sI - A)X(s) = x(0) + BU(s) \quad (33)$$

where I is a unit matrix of the same order as A. Without introducing I, (S-A) is not defined, for s is a scalar and A is an $n \times n$ matrix. The pre-multiplication of $(sI-A)^{-1}$ to (33) gives

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \quad (34)$$

This response consists of the zero-input response (the response due to nonzero x(0)) and the zero-state response (the response due to nonzero U(t)). Substituting X(s) into the Laplace transform of (24), we obtain

$$Y(s) = C(sI - A)^{-1}x(0) + [C(sI - A)^{-1}B + D]U(s) \quad (35)$$

From (35), D = 0 and also x(0) = 0. Therefore, we can write that

$$Y(s) = C(sI - A)^{-1}BU(s) \quad (36)$$

However,

$$(sI - A) = \begin{bmatrix}s & -1 & 0 & 0\\ -3\omega^2 & s & 0 & -2R_o\omega\\ 0 & 0 & s & -1\\ 0 & \frac{2\omega}{R_o} & 0 & s\end{bmatrix} \quad (37)$$

The determinant det of $(sI - A)$ is

$$det\big((sI - A)\big) = s^4 + \omega^2 s^2 \quad (38)$$

Also the adjoin, Adj of $(sI - A)$ is written as

$$Adj = \begin{bmatrix}(s^3 + 4\omega^2 s) & (-s^2) & (0) & (2R_o\omega s)\\ (3\omega^2 s^2) & (s^3) & (0) & (2R_o\omega s^2)\\ \left(-\frac{6\omega^3}{R_o}\right) & \left(-\frac{2\omega s}{R_o}\right) & (s^3 + \omega^2 s) & -(s^2 + 3\omega^2)\\ \left(-\frac{6\omega^3 s}{R_o}\right) & \left(-\frac{2\omega s^2}{R_o}\right) & (0) & (s^3 - 3\omega^2 s)\end{bmatrix} \quad (39)$$

Let

$$adj(sI-A)=N=\begin{bmatrix}N_{11} & N_{12} & N_{13} & N_{14}\\ N_{21} & N_{22} & N_{23} & N_{24}\\ N_{31} & N_{32} & N_{33} & N_{34}\\ N_{41} & N_{42} & N_{43} & N_{44}\end{bmatrix} \quad (40)$$

Also

$$B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\\ B_{31} & B_{32}\\ B_{41} & B_{42}\end{bmatrix} \quad (41)$$

However

$$(sI-A)^{-1}=\frac{adj(sI-A)}{det(sI-A)} \quad (42)$$

Therefore, (36) becomes

$$Y(s)=C\frac{1}{det(sI-A)}adj(sI-A)BU(s) \quad (43)$$

$$adj(sI-A)B=\begin{bmatrix}N_{11} & N_{12} & N_{13} & N_{14}\\ N_{21} & N_{22} & N_{23} & N_{24}\\ N_{31} & N_{32} & N_{33} & N_{34}\\ N_{41} & N_{42} & N_{43} & N_{44}\end{bmatrix}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\\ B_{31} & B_{32}\\ B_{41} & B_{42}\end{bmatrix} \quad (44)$$

$$adj(sI-A)B=\begin{bmatrix}N_{12}B_{21} & N_{14}B_{42}\\ N_{22}B_{21} & N_{24}B_{42}\\ N_{32}B_{21} & N_{34}B_{42}\\ N_{42}B_{21} & N_{44}B_{42}\end{bmatrix} \quad (45)$$

$$C[adj(sI-A)B]=CNB$$
$$=\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}N_{12}B_{21} & N_{14}B_{42}\\ N_{22}B_{21} & N_{24}B_{42}\\ N_{32}B_{21} & N_{34}B_{42}\\ N_{42}B_{21} & N_{44}B_{42}\end{bmatrix}$$
$$=\begin{bmatrix}N_{32}B_{21} & N_{34}B_{42}\end{bmatrix} \quad (46)$$

$$N_{32}B_{21}=\left(-\frac{2\omega s}{R_o}\right)\left(\frac{1}{m}\right)=-\frac{2\omega s}{mR_o} \quad (47)$$

$$N_{34}B_{42}=-(s^2+3\omega^2)\left(\frac{1}{mR_o}\right)=\frac{-(s^2+3\omega^2)}{mR_o} \quad (48)$$

Therefore, (43) becomes

$$Y(s)=\frac{1}{det(sI-A)}\begin{bmatrix}N_{32}B_{21} & N_{34}B_{42}\end{bmatrix}U(s) \quad (49)$$

$$Y(s)=\frac{1}{s^4+\omega^2s^2}\begin{bmatrix}-\frac{2\omega s}{mR_o} & \frac{-(s^2+3\omega^2)}{mR_o}\end{bmatrix}U(s) \quad (50)$$

$$Y(s)=\begin{bmatrix}\frac{-2\omega s}{mR_o(s^4+\omega^2s^2)} & \frac{-(s^2+3\omega^2)}{mR_o(s^4+\omega^2s^2)}\end{bmatrix}U(s) \quad (51)$$

$$Y(s)=\begin{bmatrix}\frac{-\frac{2\omega s}{mR_o}}{s^2(s^2+\omega^2)} & \frac{-\frac{(s^2+3\omega^2)}{mR_o}}{s^2(s^2+\omega^2)}\end{bmatrix}U(s) \quad (52)$$

Considering the small variations in X(S), Y(S), and U(S), (52) becomes

$$\Delta Y(s)=\begin{bmatrix}\frac{-j\frac{2\omega^2}{mR_o}}{s^2(s^2+\omega^2)} & \frac{-\frac{2\omega^2}{mR_o}}{s^2(s^2+\omega^2)}\end{bmatrix}\Delta U(s) \quad (53)$$

$$\Delta Y(s)=\left[\left(-j\frac{2}{mR_o}\right)\left(\frac{\omega^2}{s^2(s^2+\omega^2)}\right) \quad \left(-\frac{2}{mR_o}\right)\left(\frac{\omega^2}{s^2(s^2+\omega^2)}\right)\right]\Delta U(s) \quad (54)$$

The expression in (54) is the small-signal representation of a satellite in geostationary orbit. Further information, such as transfer function of satellite orbit can be obtained.

## IV. Discussion

Representing the governing differential equations by first order state equations makes it possible to directly solve the state equations in time, using standard numerical methods and efficient algorithms on today's fast digital computers. Since the state equations are always of first order irrespective of the system's order or the number of inputs and outputs, the greatest advantage of state-space methods is that they do not formally distinguish between single-input, single-output systems and multivariable systems, allowing efficient design and analysis of multivariable systems with the same ease as for single variable systems. Furthermore, using state-space methods it is possible to directly design and analyze nonlinear satellite system which is utterly impossible using classical methods. When dealing with linear satellite systems, state-space methods result in repetitive linear algebraic manipulations (such as matrix multiplication, inversion, solution of a linear matrix equation, etc.), which are easily programmed on a digital computer.

From the small-signal expression of (54), the satellite orbit transfer function which is a two-element matrix, can be seen to be dependent on the satellite mass, orbital radius $R_o$, and angular velocity ω.

The sweep angle $\beta_o$, mass of earth $M_E$, and satellite propulsion $f_{r_o}$ and $f_{\beta_o}$ in $r_o$ and $\beta_o$ directions do not directly reflect in the small-signal expression.

## V. Conclusion

In conclusion, the control small-signal nature of Keplarian satellite orbit has been theoretically investigated in terms of characteristic equation. The obtained small-signal expression represents the nonlinear description of the satellite orbit. This linearized expression will provide a significant opportunity for further critical stability analysis of the satellite orbit in subsequent work.

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